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The Büttiker–Landauer Model Generalized

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Büttiker and Landauer studied scattering off an oscillating rectangular barrier in order to shed light on the time aspects of tunneling. The expression for the traversal time resulting from this study is controversial. In addition, doubts have recently been expressed on technical aspects of their work. In an attempt to clarify these issues, we investigate a generalization of their model to arbitrary oscillating barriers, $V(x, t) = V_0(x) + V_1(x) \cos \omega t$. In the process, we confirm that Büttiker and Landauer's work is technically sound. However, we show, by several examples, that no direct general relation exists between the characteristic frequency of an oscillating barrier and the duration of the tunneling process. For a wide range of realistic parameters this characteristic frequency does not even exist.

KEY WORDS: Tunneling; oscillating barriers; duration of tunneling; characteristic frequency.

1. INTRODUCTION

During the last decade, new techniques, such as molecular beam epitaxy (MBE), have made possible the fabrication of essentially one-dimensional semiconductor structures on a nanometer scale. The potential device applications of configurations in which tunneling processes play a crucial role (see, e.g., ref. 1) have given urgency to the need for a reliable kinetic theory for the charge carriers in such systems. Basic to a kinetic theory of this kind is an understanding of the fundamental processes involved. In particular, one would like to know how long the tunneling process takes, and what the characteristic frequency is when tunneling particles couple to

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This paper is dedicated to E. G. D. Cohen.

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phonons, which typically constitute the heat bath in this context. These questions, elementary as they may seem, have been controversial in recent years.

As a device for studying the time aspects of tunneling, Büttiker and Landauer introduced[2,3] the following simple model: A plane wave with energy E impinges on a rectangular barrier on the x interval $(-d/2, d/2)$ in one dimension. To the constant barrier height V_0 is added an oscillating component $V_1 \cos \omega t$. The particles interacting with such a barrier can absorb or emit modulation quanta $\hbar\omega$. One can think of this model as a rough caricature of tunneling electrons coupled in the barrier to optical phonons of fixed frequency. Natural questions are: (i) What are the amplitudes $A_{\pm 1}(\omega)$ and $B_{\pm 1}(\omega)$ of the first sidebands (with energies $E \pm \hbar\omega$) for transmitted and reflected particles? (ii) What is the characteristic frequency separating the low-frequency regime, when particles essentially "see" the instantaneous barrier, from the high-frequency regime, when large deviations from this picture manifest themselves? (iii) What conclusions can be drawn about the duration of the tunneling process on the basis of the results found? Büttiker and Landauer[2–4] answered the first question for a general rectangular barrier, and discussed the second explicitly for opaque ones. Finally, on the basis of their results, they proposed that the duration of the tunneling process is given by the first sideband amplitudes in the $\omega \to 0$ limit. For easy reference, some of Büttiker and Landauer's results are collected in Appendix A.

In the recent controversy over tunneling times,[3] the intriguing results of Büttiker and Landauer have played a central role. They raise two types of problems. The first type is purely technical: Does it make sense to expand in what seems to be $V_1/\hbar\omega$, retain only the first-order terms, and take seriously the resulting $\omega \to 0$ limit? Is the treatment of the time-dependent problem at all consistent?[5]

We shall give reassuring answers to these questions. For the *generalized* oscillating barrier $V(x, t) = V_0(x) + V_1(x) \cos \omega t$ where $V_0(x)$ and $V_1(x)$ are arbitrary functions on the interval $(-d/2, d/2)$, we show in Section 2 how to obtain the $\omega \to 0$ limit directly. For finite-frequency results, one is, however, forced to go back to the perturbation scheme introduced by Büttiker and Landauer.[3] The mechanics of this scheme is investigated in Appendix B, and it is found to be well behaved. In addition, the $\omega \to 0$ limit is studied to $O(V_1)$. It is shown to coincide with that obtained directly in Section 2 for arbitrary barriers. The $O(\omega)$ terms are, in general, very complicated, and will only be briefly commented upon.

The second type of problem raised by Büttiker and Landauer's work is one of interpretation. In a separate publication,[6] we have critically

[3] See refs. 4–6 for reviews with extensive lists of references.

reviewed the various proposals for tunneling times and discussed their interrelations. In the present paper we shall restrict ourselves to scrutiny of oscillating barriers and their bearing on the tunneling time issue. The general connection between the complex "times" introduced by Sokolovski and Baskin[(7)] and the sidebands of oscillating barriers is given in Section 2. In Section 3 we explore the proposed relationship between the characteristic frequency separating the low- and high-frequency regimes and the duration of the tunneling process. As a contrast to the opaque barriers studied in refs. 2 and 3, we investigate transparent ones. The results show that the "times" characterizing the low-frequency amplitudes are, in general, *different* from those governing the sideband asymmetry. For the special case of an oscillating δ-function barrier, the complete ω dependence of the sideband amplitudes is determined to $O(V_1)$. A finite characteristic frequency is found. Since the duration of the tunneling process clearly vanishes in this case, the example shows that no simple relation exists between the duration of tunneling and the characteristic frequency of oscillating barriers.

With realistic parameters in a GaAs/AlGaAs/GaAs structure we find that, except for very thin barriers, the characteristic frequency ω_c is only well defined for barrier widths $d \gtrsim \hbar[2(V_0 - E)/m]^{1/2}/E$. This means that ω_c is not defined for most barriers of practical interest.

We close by a summary, and some comments on the Büttiker–Landauer times.

2. SCATTERING OFF AN ARBITRARY OSCILLATING BARRIER

In this section we generalize Büttiker and Landauer's work on oscillating barriers to barriers of arbitrary shape,[4] $V(x, t) = V_0(x) + V_1(x) \cos \omega t$ on the interval $(-d/2, d/2)$ (see Fig. 1). Section 2.1 contains the basics, whereas a formal treatment of the $\omega \to 0$ limit is presented in Section 2.2. A detailed discussion of the perturbation scheme, valid for arbitrary ω, is relegated to Appendix B. In Section 2.3 the connection to Sokolovski and Baskin's work is pointed out.

2.1. Basics

An incoming particle of energy E interacting with an oscillating barrier can absorb or emit modulation quanta $\hbar\omega$. The solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = H(x, t)\, \Psi(x, t) \tag{2.1}$$

[4] One could also generalize to a set of frequencies ω_i in a straightforward manner, but from our present perspective the results would not be particularly illuminating. Note also that the phase of $\cos \omega t$ is arbitrary. We choose it as zero for convenience.

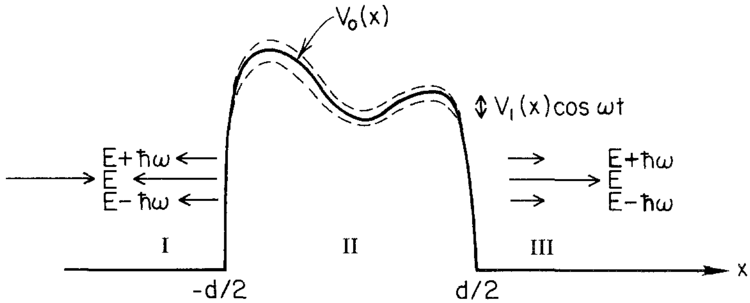

Fig. 1. The arbitrary scattering potential $V_0(x)+V_1(x)\cos\omega t$. The solid line represents the static part $V_0(x)$ and the dashed lines represent the perturbation $V_1(x)\cos\omega t$. The potential is confined to region II, $|x|<d/2$. A particle incident from the left with energy E is reflected or transmitted at energies $E\pm n\hbar\omega$. The figure shows the first sidebands only.

for a plane wave scattered off an arbitrary oscillating barrier must therefore have the general form[3]

$$\Psi(x,t)=\sum_n \phi_n(x)\, e^{-(i/\hbar)(E+n\hbar\omega)t} \tag{2.2}$$

The stationary solutions ϕ_n at energy $E+n\hbar\omega$ must be plane waves in regions I and III of Fig. 1 (outgoing waves only, except when $n=0$). The form of the solutions in region II must be determined in successive approximations. We write it as

$$\phi_n(x)=C_n\Xi_n(x)+D_n\Gamma_n(x) \tag{2.3}$$

where, to zeroth order in V_1, $\Xi_n(x)$ and $\Gamma_n(x)$ are eigenfunctions of the Hamiltonian $H_0=-\hbar^2/2m(\partial^2/\partial x^2)+V_0(x)$. With (2.2), the Schrödinger equation (2.1) reads

$$\begin{aligned}&\sum_n (E+n\hbar\omega)\,\phi_n(x)\, e^{-(i/\hbar)(E+n\hbar\omega)t}\\&\quad=\sum_n\left[H_0+\frac{V_1(x)}{2}(e^{i\omega t}+e^{-i\omega t})\right]\phi_n(x)\, e^{-(i/\hbar)(E+n\hbar\omega)t}\end{aligned} \tag{2.4}$$

Since this equation must hold at all times, we have

$$[H_0-(E+n\hbar\omega)]\,\phi_n=-\tfrac{1}{2}V_1(x)[\phi_{n+1}+\phi_{n-1}] \tag{2.5}$$

Similarly, the standard boundary conditions (of continuity of the wave function and its derivative at $x=\pm d/2$) must hold for each component ϕ_n separately. This determines the amplitudes B_n and A_n of the plane wave solutions in regions I and III, and the amplitudes C_n and D_n of (2.3) in region II.

The infinite set of coupled, time-independent equations (2.5) and the corresponding matching conditions are thus seen to be straightforward consequences of the time-dependent Schrödinger equation (contrary to claims made in ref. 5). They mildly generalize the equations used by Büttiker and Landauer,[3] and form the basis of our work.

2.2. The $\omega \to 0$ Limit

In order to discuss the $\omega \to 0$ limit of the problem at hand, it is convenient essentially to reverse the step of (2.2) by introduction of the discrete Fourier transform

$$\Phi(x, s) = \sum_n \phi_n(x)\, e^{-ins} \tag{2.6}$$

This turns (2.5) into

$$[H_0 + V_1(x) \cos s - E]\, \Phi(x, s) = i\hbar\omega \frac{\partial}{\partial s} \Phi(x, s) \tag{2.7}$$

Since substitution of $s = \omega t$ confirms that (2.7) is equivalent to the original time-dependent Schrödinger equation (2.1), this is not particularly useful in general.

However, when the $\omega \to 0$ limit is taken on (2.7), the right-hand side vanishes, and one is left with a *stationary* scattering problem at energy E and with the potential $V(x) = V_0(x) + V_1(x) \cos s$. The corresponding solution $\Phi^{(0)}$ (where the superscript refers to the $\omega \to 0$ limit) can be viewed as a *functional* of the form of the barrier (in addition to depending explicitly on x, of course):

$$\Phi^{(0)}(x, s) = \phi_0[V_0(x) + V_1(x) \cos s] \tag{2.8}$$

The inverse transform is $(n \geqslant 0)$

$$\begin{aligned}
\phi_{\pm n}(x) &= \int_{-\pi}^{\pi} \frac{ds}{2\pi}\, e^{\pm ins} \phi_0[V_0(x) + V_1(x) \cos s] \\
&= \int_{-\pi}^{\pi} \frac{ds}{2\pi} \cos ns\, \phi_0[V_0(x) + V_1(x) \cos s] \\
&= \int_{-d/2}^{d/2} dx_1 \cdots \int_{-d/2}^{d/2} dx_n \frac{V_1(x_1) \cdots V_1(x_n)}{2^n n!} \frac{\delta^n \phi_0[V_0(x)]}{\delta V(x_1) \cdots \delta V(x_n)} \\
&\quad + O(V_1(x)^{n+2})
\end{aligned} \tag{2.9}$$

where the functional derivative can be defined as

$$\int dx\, U(x) \frac{\delta \phi_0}{\delta V(x)} = \lim_{N \to \infty} \sum_{j=1}^{N} U(x_j) \frac{\partial \phi_0}{\partial V(x_j)} \tag{2.10}$$

With our choice of phase for $\cos \omega t$, we have used that $\cos s$ is an even function of s, and as a result, the sidebands, phase factors included, are symmetric in $\pm n$ in the $\omega \to 0$ limit.

Outside the barrier region, the wave functions are the plane waves, and in the $\omega \to 0$ limit we have

$$\phi^{\mathrm{I}}_{\pm n}(x) = B^{(0)}_{\pm n} e^{-ikx} + \delta_{n,0} e^{ikx} \tag{2.11}$$

$$\phi^{\mathrm{III}}_{\pm n}(x) = A^{(0)}_{\pm n} e^{ikx} \tag{2.12}$$

Thus, (2.9) directly gives the sideband amplitudes for transmitted particles in the $\omega \to 0$ limit as

$$\begin{aligned} A^{(0)}_{\pm n} &= \int_{-\pi}^{\pi} \frac{ds}{2\pi} \cos ns\, A_0[V_0(x) + V_1(x) \cos s] \\ &= \int_{-d/2}^{d/2} dx_1 \cdots \int_{-d/2}^{d/2} dx_n \frac{V_1(x_1) \cdots V_1(x_n)}{2^n n!} \frac{\delta^n A_0[V_0(x)]}{\delta V(x_1) \cdots \delta V(x_n)} \\ &\quad + O(V_1(x)^{n+2}) \end{aligned} \tag{2.13}$$

The result for the reflected sidebands is analogous, with B replacing A.

The above procedure for obtaining the $\omega \to 0$ limit is essentially a formal version of the intuitive argument used in ref. 3 for the rectangular barrier. It has the advantage of producing compact general results in a straightforward manner. Explicit results for the special case of an oscillating δ-function barrier are found in Section 3.

For results beyond the $\omega \to 0$ limit, one must return to the set of equations (2.5). Since it is the oscillating part of the barrier, $V_1(x)$, which couples these equations, it is natural to try a perturbation series in powers of V_1. This scheme, introduced in ref. 3 for rectangular barriers, is studied in Appendix B. There we construct a general proof that the inhomogeneity in equations of the form (2.5), at every level of approximation, does not contain solutions of the corresponding homogeneous equation. This constitutes a proof that the perturbation scheme is well behaved. Furthermore, even though the dimensionless parameter appears[3] to be $V_1/\hbar\omega$, in reality it is V_1/E or $V_1/(V_0 - E)$, and we show that the $\omega \to 0$ limit can be taken on the first-order terms, and give results in agreement with (2.9)–(2.13), to $O(V_1)$.

2.3. Connection to a Complex "Time"

Recently, Sokolovski and Baskin[7] proposed a formal generalization to the quantum domain of the classical time $\tau_{\rm cl}^{\Omega}$ spent in a region of space Ω. For one-dimensional motion, they wrote the classical time as

$$\tau_{\rm cl}^{\Omega}[x(t)] = \int_{t_i}^{t_f} dt \int_{\Omega} dx\, \delta(x - x(t)) \tag{2.14}$$

where $x(t)$ is the classical path from the initial point $x_i(t_i)$ to the final one $x_f(t_f)$. They proposed the quantum generalization of (2.14) as the path integral average (see, e.g., ref. 8)

$$\tau^{\Omega}(x_i, t_i; x_f, t_f) = \langle \tau_{\rm cl}^{\Omega}[x(\cdot)] \rangle_{\rm paths} \tag{2.15}$$

Sokolovski and Baskin showed that τ^{Ω} is, in general, complex, and stressed the fact that it is a nonunique generalization of the classical time. In our opinion, their construct can only be given physical meaning through connections to concrete model situations. In ref. 7, one example was offered: The real and imaginary parts of τ^{Ω} are closely related to local Larmor "times."[6,9–11]

Here we point out that the connection[7] between τ^{Ω} and the oscillating barrier is quite general. In ref. 7, Sokolovski and Baskin gave the following expressions for their complex times in the context of particles with fixed energy E transmitted or reflected by a barrier,

$$\tau_T^{\Omega} = i\hbar \int_{\Omega} dx \frac{\delta \ln A_0}{\delta V(x)}; \qquad \tau_R^{\Omega} = i\hbar \int_{\Omega} dx \frac{\delta \ln B_0}{\delta V(x)} \tag{2.16}$$

where Ω stands for the barrier interval $(-d/2, d/2)$. Comparison with (2.13) shows that, with $V_1(x) = V_1 = \text{const}$, but for general $V_0(x)$, one has

$$\frac{A_{\pm 1}^{(0)}}{A_0} = -i \frac{V_1}{2\hbar} \tau_T^{\Omega}; \qquad \frac{B_{\pm 1}^{(0)}}{B_0} = -i \frac{V_1}{2\hbar} \tau_R^{\Omega} \tag{2.17}$$

Thus, the complex "times" τ_T^{Ω} and τ_R^{Ω} have a direct physical interpretation as relative sideband amplitudes for the oscillating barrier in the $\omega \to 0$ limit.

3. THE CHARACTERISTIC FREQUENCY

3.1. The Opaque Barrier

Even for a rectangular barrier to $O(V_1)$, the results[3] (A.1) and (A.2) for the sideband amplitudes are sufficiently complicated that a characteristic frequency separating the high- and low-frequency regimes is not immediately apparent. However, for the important case of opaque barriers, i.e., when $\exp(-2\kappa d) \ll 1$ (with $\hbar^2\kappa^2/2m = V_0 - E$), the relative intensities of the transmitted sidebands simplify to[3]

$$I^T_{\pm 1}(\omega) = \left|\frac{A_{\pm 1}}{A_0}\right|^2 = \left(\frac{V_1}{2\hbar\omega}\right)^2 [\exp(\pm\omega\tau_T^{\mathrm{BL}}) - 1]^2 \tag{3.1}$$

where $\tau_T^{\mathrm{BL}} = md/\hbar\kappa$ (and it has been assumed that $\hbar\omega$ is small compared to E and $V_0 - E$). The characteristic frequency is clearly $\omega_c = 1/\tau_T^{\mathrm{BL}}$ in this case. It determines the low-frequency limit through

$$I^T_{\pm 1}(0) = \left(\frac{V_1 \tau_T^{\mathrm{BL}}}{2\hbar}\right)^2 \tag{3.2}$$

and also the sideband asymmetry

$$F(\omega) = \frac{I^T_{+1} - I^T_{-1}}{I^T_{+1} + I^T_{-1}} = \tanh \omega\tau_T^{\mathrm{BL}} \tag{3.3}$$

For general barriers, Büttiker and Landauer[3] take (3.2) as the *definition* of the time τ^{BL} (here: for transmission). We shall call τ_T^{BL} and τ_R^{BL} the Büttiker–Landauer times.

In the sense defined by (3.1)–(3.3), τ_T^{BL} is clearly *the* characteristic time here. The question remains whether this time must, or can, be interpreted as the duration of the tunneling process. In order to answer this question, we shall consider some other simple cases which can be discussed explicitly to varying degree.

On the basis of the general results[3] for rectangular barriers quoted in Appendix A, it is clearly possible to expand $I^T_{\pm 1}(\omega)$ and the relative sideband intensities for reflected particles $I^R_{\pm 1}(\omega)$ to $O(\omega)$. Even for rectangular barriers this is a tedious task, with complicated results. Before proceeding to other cases, we quote the result for reflection from opaque barriers:

$$I^R_{\pm 1}(\omega) = \left(\frac{V_1 \tau_R^{\mathrm{BL}}}{2\hbar}\right)^2 \left(1 \pm \frac{1}{2}\omega\tau_\kappa\right) \tag{3.4}$$

where

$$\tau_R^{\mathrm{BL}} = \frac{2mk}{\hbar\kappa(\kappa^2 + k^2)} = \frac{\hbar}{V_0}\left(\frac{E}{V_0 - E}\right)^{1/2}; \qquad \tau_\kappa = \frac{2m}{\hbar\kappa^2} = \frac{\hbar}{V_0 - E} \tag{3.5}$$

3.2. Transparant Barriers

We now turn from opaque barriers to the opposite extreme, transparant ones, for which $\kappa d \ll 1$. This case covers two distinct possibilities (see Fig. 2). Either the barrier is *thin*, and κd and kd are small for all energies, $E < V_0$. Or κ is small, i.e., the energy almost equals the barrier height. The quantity kd should then be treated as of $O(1)$. We shall give the relative sideband intensities for the latter case. Results for thin barriers can be obtained from these by keeping lowest order terms in kd only.

Straightforward but tedious expansion of (A.1) and (A.2) with $\kappa d \ll 1$ and $kd = O(1)$ gives

$$I^T_{\pm 1}(\omega) = \left(\frac{V_1}{2\hbar}\frac{md}{\hbar k}\right)^2 \frac{1 + \frac{2}{3}\eta^2 + (1/36)\eta^4}{1 + \frac{1}{4}\eta^2} [1 \mp \omega\tau_k \theta(\eta)]$$

$$\theta(\eta) = \frac{1 + \frac{2}{3}\eta^2 + (1/90)\eta^4 - (13/360)\eta^6 - (7/4320)\eta^8}{1 + (11/12)\eta^2 + (7/36)\eta^4 + (1/144)\eta^6} \tag{3.6}$$

$$I^R_{\pm 1}(\omega) = \left(\frac{V_1}{2\hbar}\frac{2m}{\hbar(\kappa^2 + k^2)}\right)^2 \frac{1 + \frac{1}{3}\eta^2 + \frac{1}{9}\eta^4}{1 + \frac{1}{4}\eta^4} [1 \mp \omega\tau_k \rho(\eta)]$$

$$\rho(\eta) = \frac{1 + \frac{1}{2}\eta^2 + (41/360)\eta^4 + (1/180)\eta^6 - (1/270)\eta^8}{1 + (7/12)\eta^2 + (7/36)\eta^4 + (1/36)\eta^6} \tag{3.7}$$

where $\eta = kd$ and $\tau_k = \hbar/E$. The factors $\theta(kd)$ and $\rho(kd)$, which are exact consequences of (A.1) and (A.2), are correcting τ_k as the characteristic time associated with the sideband asymmetries of (3.6) and (3.7). These factors are shown in Fig. 3. For comparison, the factor $\frac{1}{2}kd$, correcting τ_k as the characteristic time $\tau_T^{\mathrm{BL}} = md/\hbar k$ in the sideband amplitude in (3.6), is also shown. In (3.7) the sideband amplitude is determined by τ_k for "all" k (i.e.,

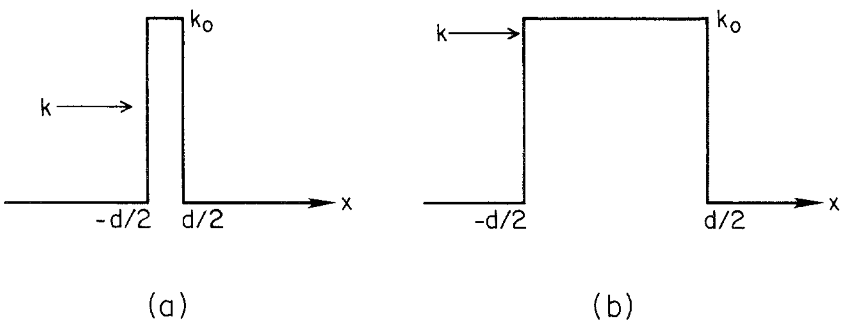

Fig. 2. Different limits for scattering off transparent rectangular barriers. (a) $kd \ll 1$. (b) $kd \sim O(1)$.

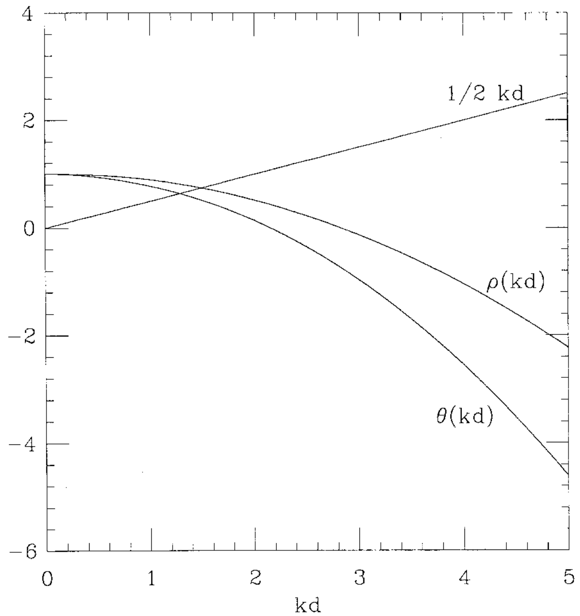

Fig. 3. The factors $\theta(kd)$ and $\rho(kd)$ of Eqs. (3.6) and (3.7), giving the corrections to τ_k as the characteristic time associated with the sideband asymmetries in the case of transparent barriers. The factor $\frac{1}{2}kd$, correcting τ_k as the characteristic time $\tau_T^{\mathrm{BL}} = md/\hbar k$, in $I_{\pm 1}^T(0)$, is also shown.

to the extent that $\kappa \ll k$). These results for thin barriers indicate, as do (3.4) and (3.5), that no simple general relation exists between the time constants τ_T^{BL} and τ_R^{BL}, as found from the $\omega \to 0$ limits in analogy with (3.2), and the characteristic times associated with the sideband asymmetry. As is apparent from (3.6) and (3.7), even the *sign* of this asymmetry changes. Figure 3 shows that $\theta(kd)$ and $\rho(kd)$ go negative at $kd \simeq 2.1$ and $kd \simeq 2.8$, respectively. Note, furthermore, that the Büttiker–Landauer time for transmission in (3.6) tends to zero with d as $d \to 0$. However, τ_R^{BL} of (3.7) tends to $\hbar/V_0$ as $d \to 0$, in conflict with an interpretation of τ_R^{BL} as the duration of a tunneling process.

3.3. The δ-Function Barrier

Except for transmission through an opaque barrier, where the full frequency dependence (within the range where $\hbar\omega$ is small relative to E and $V_0 - E$) was given by (3.1), all these cases were examined to $O(\omega)$ only. For an unambiguous determination of a characteristic frequency, this is clearly not sufficient. We therefore turn to the oscillating δ-function barrier, for which the entire frequency dependence of the sideband intensities can be calculated explicitly. As an interesting contrast, Elberfeld and Kleber[12] have recently constructed exact results for time-dependent scattering off a stationary δ-function.

Starting from the rectangular barrier on the interval $(-d/2, d/2)$, with oscillating height $V_0 + V_1 \cos \omega t$, we let $d \to 0$, $V_0 \to \infty$, and $V_1 \to \infty$ in such a way that

$$V_0 d = \hbar c_0; \qquad V_1 d = \hbar c_1 \tag{3.8}$$

with c_0 and c_1 (for convenience chosen with the dimension of velocity) kept constant. The Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \hbar(c_0 + c_1 \cos \omega t)\, \delta(x) \right] \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t) \tag{3.9}$$

In this limiting case, the boundary conditions are as follows.

1. Continuity of the wave function in $x = 0$:

$$\lim_{\varepsilon \to 0} \Psi(-\varepsilon, t) = \lim_{\varepsilon \to 0} \Psi(\varepsilon, t) = \Psi(0, t) \tag{3.10}$$

2. A finite jump in the derivative of Ψ in $x = 0$:

$$\lim_{\varepsilon \to 0} \left(\frac{\partial \Psi(\varepsilon, t)}{\partial x} - \frac{\partial \Psi(-\varepsilon, t)}{\partial x} \right) = \frac{2m}{\hbar} (c_0 + c_1 \cos \omega t)\, \Psi(0, t) \tag{3.11}$$

As always, modulation quanta $\hbar\omega$ can be absorbed or emitted in the barrier. The wave function must therefore have the form

$$\text{for } x < 0: \quad \Psi_<(x, t) = \exp\left(ikx - \frac{i}{\hbar} Et \right) + \sum_n B_n \exp\left(-ik_n x - \frac{i}{\hbar} E_n t \right) \tag{3.12}$$

$$\text{for } x > 0: \quad \Psi_>(x, t) = \sum_n A_n \exp\left(ik_n x - \frac{i}{\hbar} E_n t \right) \tag{3.13}$$

with $E_n = E + n\hbar\omega$, and $k_n = k(1 + n\hbar\omega/E)^{1/2}$. The continuity condition (3.10) immediately gives the B_n in terms of the A_n:

$$1 + B_0 = A_0 \tag{3.14}$$

$$B_n = A_n \qquad (n \neq 0) \tag{3.15}$$

The condition on the derivative (3.11) gives

$$\sum_n ik_n A_n e^{-(i/\hbar) E_n t} - ike^{-(i/\hbar) Et} + \sum_n ik_n B_n e^{-(i/\hbar) E_n t}$$
$$= \frac{2m}{\hbar}\left[c_0 + \frac{c_1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] \sum_n A_n e^{-(i/\hbar) E_n t} \tag{3.16}$$

From these equations we obtain an infinite set of coupled equations for the amplitudes A_n:

$$ik_n(A_n - \delta_{n,0}) = \frac{m}{\hbar}\left[c_0 A_n + \frac{c_1}{2}\left(A_{n+1} + A_{n-1}\right)\right] \tag{3.17}$$

Define the discrete Fourier transforms

$$\alpha(s) = \sum_n A_n e^{-ins} \tag{3.18}$$

$$\beta(s) = \sum_n k_n A_n e^{-ins} \tag{3.19}$$

With (3.18) and (3.19), Eq. (3.17) transforms to

$$i\beta(s) = \alpha(s)\frac{m}{\hbar}\left(c_0 + c_1 \cos s\right) + ik \tag{3.20}$$

Even for the oscillating δ-function barrier a complete solution of the set of equations (3.17), or equivalently, (3.20), seems difficult. However, we are mostly interested in two cases: (i) The $\omega \to 0$ limit to all orders in c_1, which serves as an illustration of the results in Section 2. (ii) The full ω dependence of the first sidebands, to $O(c_1)$.

When $\omega \to 0$, (3.18) and (3.19) give $\beta^{(0)}(s) = k\alpha^{(0)}(s)$ and (3.20) then yields

$$\alpha^{(0)}(s) = \left(1 + i\frac{c_0}{v} + i\frac{c_1}{v}\cos s\right)^{-1} \tag{3.21}$$

Inverse Fourier transformation gives (introduce $z = e^{is}$ and use the calculus of residues)

$$A^{(0)}_{\pm n} = \int_{-\pi}^{\pi} \frac{ds}{2\pi}\,\alpha^{(0)}(s)\cos ns$$
$$= \frac{1}{1 + ic_0/v}\,\frac{\xi^{-n}[1 - (1 - \xi^2)^{1/2}]^n}{(1 - \xi^2)^{1/2}} \tag{3.22}$$

with $\xi = c_1/(iv - c_0)$. Since $\xi \neq \pm 1$ always, this solution is valid for any c_1. Equation (3.22) contains, for example, the depleted intensity at the original energy, $|A_0^{(0)}|^2 = [1 + (c_0/v)^2 + (c_1/v)^2]^{-1}$, to all orders in c_1, and the leading sideband intensities $\sim c_1^{2n}$.

More interesting from our present perspective is the complete ω dependence of $A_{\pm 1}$, to $O(c_1)$. From (3.17) one readily derives that

$$I^T_{\pm 1}(\omega) = \left|\frac{A_{\pm 1}}{A_0}\right|^2 = \frac{c_1^2}{4(v^2 + c_0^2)}\left(1 \pm \frac{\hbar\omega}{E + \frac{1}{2}mc_0^2}\right)^{-1} \tag{3.23}$$

$$I^R_{\pm 1}(\omega) = \left|\frac{B_{\pm 1}}{B_0}\right|^2 = \left(\frac{v}{c_0}\right)^2 I^T_{\pm 1}(\omega) \tag{3.24}$$

The time τ_T^{BL} defined from the $\omega \to 0$ limit of $I^T_{\pm 1}(\omega)$ vanishes with d, as one would expect for the duration of a tunneling process. On the other hand, the sideband asymmetry follows from (3.23) as

$$F(\omega) \equiv \frac{I^T_{+1} - I^T_{-1}}{I^T_{+1} + I^T_{-1}} = -\frac{\hbar\omega}{E + \frac{1}{2}mc_0^2} \tag{3.25}$$

The characteristic frequency associated with (3.25) is clearly finite. In spite of what the result (3.1) for transmission through opaque barriers might lead one to believe, the example of the oscillating δ-function barrier shows unambiguously that no direct relation exists between the characteristic frequency and the duration of tunneling.

3.4. Physical Interpretation and Existence of a Characteristic Frequency

Büttiker and Landauer introduced ω_c as an estimate of the frequency at which the asymmetry in the sideband intensities becomes appreciable. In order to quantify this concept, we define ω_c as

$$|F(\omega_c)| = \tanh 1 = 0.76\ldots \tag{3.26}$$

In taking the absolute value of F, one can use the same definition for thick and thin barriers [see (3.3) and (3.25)]. Clearly, the above definition is not the only possible one. Nothing dramatic happens with the asymmetry at $\omega = \omega_c$. We have chosen (3.26) since it is the definition which yields $\omega_c^{-1} = \tau_T^{\mathrm{BL}}$ (asymptotically) for thick barriers, in agreement with Büttiker and Landauer's result. However, changing the definition quantitatively does not affect the qualitative points we want to make in the following.

The physics which determines the asymmetry also determines ω_c. As pointed out by Büttiker and Landauer, in the case of opaque barriers

ω_c is completely determined by the energy sensitivity of the tunneling probability $T(E) = |A_0(E)|^2$. In the opposite extreme, the δ-function result (3.25) shows that, when $c_0 \to 0$, no property of the barrier itself remains. The sideband asymmetry is then determined by the density of states alone. The minus sign in (3.25) reflects this dependence, since the density of states $\sim E^{-1/2}$ in that case.

In going from thick to thin barriers, the asymmetry changes sign. One can therefore expect an intermediate region where the asymmetry depends only weakly on ω. In this region the sideband asymmetry will be small even for $\hbar\omega \gtrsim E$. Since $I^T_{-1} = 0$ for $\hbar\omega > E$,[13] a definition of a characteristic frequency in terms of the sideband asymmetry $F(\omega)$ is no longer possible. This is shown in Fig. 4, where we have plotted $1/\omega_c$ versus barrier width for an oscillating AlGaAs barrier between ideal GaAs leads. For comparison, $\tau_T^{\rm BL}$ is also shown. For large d, $\tau_T^{\rm BL} - \omega_c^{-1} = O(d^{-1})$. The chosen energy, $E = 0.5V_0 = 0.115$ eV, is, in fact, not very realistic. In order to have a long inelastic mean free path in the leads, the energy should be below the optical

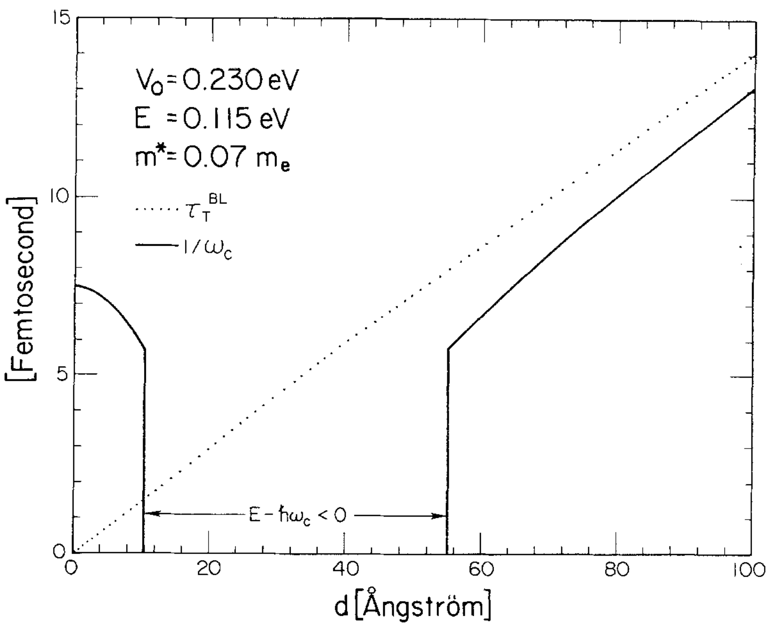

Fig. 4. Comparison between the inverse characteristic frequency ω_c^{-1} and the Büttiker–Landauer time for transmission $\tau_T^{\rm BL}$ in a GaAs/AlGaAs/GaAs structure. ω_c is defined by $|(I^T_{+1} - I^T_{-1})/(I^T_{+1} + I^T_{-1})|_{\omega=\omega_c} = \tanh 1 = 0.76...$. For our choice of energy ($E = 0.115$ eV), ω_c is not defined in the d interval (11 Å, 55 Å). Note that since E has been chosen as $\frac{1}{2}V_0$, the slope of $\tau_T^{\rm BL}$ is the same for small and large d: $m/\hbar k$ and $m/\hbar\kappa$, respectively. However, $\tau_T^{\rm BL}$ is *not* a straight line.

phonon emission edge at 0.04 eV. For $E = 0.04$ eV we find a lower bound on the barrier width of about 165 Å. This lower bound can be estimated by noting that $\hbar\omega_c \sim \hbar/\tau_T^{\mathrm{BL}} \sim \hbar^2\kappa/md$ for opaque barriers. Thus,

$$d_c \sim \frac{\hbar}{E}\left[\frac{2(V_0 - E)}{m}\right]^{1/2} \tag{3.27}$$

From Fig. 4 we see that a characteristic frequency can again be defined for very thin barriers below $d \sim 7$ Å for $E = 0.04$ eV. The interval (7 Å, 165 Å) covers essentially all barriers of practical interest. One could use a smaller value for $|F(\omega_c)|$, or consider defining ω_c on the basis of the upper sideband alone. That would reduce this interval somewhat, but the qualitative conclusion would remain: No meaningful definition of ω_c can be made for barriers of intermediate thickness.

In the limit $d \to 0$, ω_c remains finite. We therefore conclude that, even when it exists, the characteristic frequency of an oscillating barrier in general does not give the duration of the tunneling process.

4. CONCLUDING REMARKS

In this paper we have generalized the Büttiker–Landauer model to arbitrary oscillating barriers, and in the process confirmed that the methods used in ref. 3 are technically sound. We have also shown that the characteristic frequency associated with the sideband asymmetry cannot in general be interpreted as an intrinsic tunneling time.

We have not, however, discussed the merits of the Büttiker–Landauer time τ_T^{BL} determined from the low-frequency limit of the first sidebands. As is argued in a parallel review,[6] the Büttiker–Landauer time τ_T^{BL}, along with every other proposed expression for the duration of the tunneling process itself, meets some, but not all, requirements one must impose on such a time. In our opinion, the only *times* that have a precise (and complementary) meaning in this context are (i) the asymptotic phase times for transmission and reflection, which describe completed scattering events involving wave packets, and include self-interference delays in addition to tunneling times *per se*, and (ii) the dwell time, which can be defined locally, but which does not distinguish between scattering channels. For an extensive discussion of these issues, the reader is referred to ref. 6.

APPENDIX A. THE OSCILLATING RECTANGULAR BARRIER

For the rectangular oscillating barrier, Büttiker and Landauer[3] derived the following expressions for the first sidebands, to $O(V_1)$ (note that in ref. 3, A is used as amplitude for reflected particles, D for transmitted, and that overall $\pm$ signs are missing in their formulas):

$$A_{\pm 1}(\omega) = \pm \frac{V_1}{\hbar\omega} \frac{A_0(k, \kappa)}{D(k_{\pm 1}, \kappa_{\pm 1})} e^{i(k - k_{\pm 1})(d/2)} \left\{ \kappa_{\pm 1}^2 - kk_{\pm 1}) \sinh \kappa_{\pm 1} d \right.$$
$$- (\kappa^2 - kk_{\pm 1}) \frac{\kappa_{\pm 1}}{\kappa} \sinh \kappa d + i\kappa_{\pm 1}(k + k_{\pm 1})$$
$$\left. \times (\cosh \kappa d - \cosh \kappa_{\pm 1} d) \right\} \tag{A.1}$$

$$B_{\pm 1}(\omega) = \pm \frac{V_1}{\hbar\omega} \frac{A_0(k, \kappa)}{D(k_{\pm 1}, \kappa_{\pm 1})} e^{i(k - k_{\pm 1})(d/2)} \left\{ (\kappa_{\pm 1}^2 + kk_{\pm 1}) \sinh \kappa_{\pm 1} d \cosh \kappa d \right.$$
$$- (\kappa^2 + kk_{\pm 1}) \frac{\kappa_{\pm 1}}{\kappa} \cosh \kappa_{\pm 1} d \sinh \kappa d - i\kappa_{\pm 1}(k - k_{\pm 1})$$
$$\times (1 - \cosh \kappa_{\pm 1} d \cosh \kappa d)$$
$$\left. - i \left(\frac{k\kappa_{\pm 1}^2}{\kappa} - k_{\pm 1}\kappa \right) \sinh \kappa d \sinh \kappa_{\pm 1} d \right\} \tag{A.2}$$

Here $k^2 = 2mE/\hbar^2$, $k_{\pm 1}^2 = 2m(E \pm \hbar\omega)/\hbar^2$, $\kappa^2 = 2m(V_0 - E)/\hbar^2$, and $\kappa_{\pm 1}^2 = 2m(V_0 - E \mp \hbar\omega)/\hbar^2$. Furthermore,

$$D(k, \kappa) = 2(\kappa^2 - k^2) \sinh \kappa d - 4ik\kappa \cosh \kappa d \tag{A.3}$$

and $A_0 = -4ik\kappa e^{-ikd} D(k, \kappa)^{-1}$ is the transmission amplitude for the static barrier V_0. One assumes throughout that $E + \hbar\omega < V_0$ and $E - \hbar\omega > 0$.[5] For the opaque barrier, $\exp(-2\kappa d) \ll 1$, the amplitude of the transmitted sidebands is given by

$$A_{\pm 1} = \mp \frac{V_1}{2\hbar\omega} A_0 \exp\left(\mp i \frac{md}{2\hbar k} \omega \right) \left[\exp\left(\pm \omega \frac{md}{\hbar\kappa} \right) - 1 \right] \tag{A.4}$$

Here one has, in addition, assumed that $\hbar\omega$ is small compared to E and $V_0 - E$. This gives the relative sideband intensities for transmission as

$$I_{\pm 1}^T(\omega) = \left| \frac{A_{\pm 1}}{A_0} \right|^2 = \left(\frac{V_1}{2\hbar\omega} \right)^2 [\exp(\pm \omega \tau_T^{\mathrm{BL}}) - 1]^2 \tag{A.5}$$

[5] Although the formulas apply equally well for energies above V_0, with trivial changes, and for the case of dynamical localization, $E - \hbar\omega < 0$.

with $\tau_T^{\rm BL} = md/\hbar\kappa$. The low-frequency limit is

$$I_{\pm 1}^T(\omega \to 0) = \left(\frac{V_1 \tau_T^{\rm BL}}{2\hbar}\right)^2 \tag{A.6}$$

For general barriers, Büttiker and Landauer *define* $\tau_T^{\rm BL}$ by the relation (A.6). Evaluation for a general rectangular barrier yields[3]

$$\tau_T^{\rm BL} = \frac{m}{\hbar\kappa^2}\left[\frac{(\kappa^2 - k^2)\,\kappa^2 d^2 + k_0^4(1+\kappa^2 d^2)\sinh^2\kappa d + k_0^2 \kappa d(\kappa^2 - k^2)\sinh 2\kappa d}{4k^2\kappa^2 + k_0^4 \sinh^2 \kappa d}\right]^{1/2} \tag{A.7}$$

Here $k_0^2 = 2mV_0/\hbar^2 = \kappa^2 + k^2$.

APPENDIX B. THE PERTURBATION SCHEME

In this Appendix we study the perturbation series generated by (2.5), in powers of V_1 for arbitrary ω. We first show that the scheme is well behaved. Subsequently, a second expansion is made in powers of ω. The $\omega \to 0$ limit of Section 2 is recovered to $O(V_1)$. Corrections of $O(\omega)$ are discussed qualitatively.

B.1. The V_1 Expansion

In Section 2, we showed that the time-dependent Schrödinger equation decomposes into the infinite set of coupled equations (2.5):

$$[H_0 - (E + n\hbar\omega)]\,\phi_n(x) = -\tfrac{1}{2}V_1(x)[\phi_{n+1}(x) + \phi_{n-1}(x)]$$

In this Appendix we assume, for simplicity, that $V_1(x) = S(x)\,V_1$, with V_1 constant, and $S(x) = 1$ for $|x| < d/2$, and zero otherwise. [For a generalization to arbitrary $V_1(x)$, one divides the interval $(-d/2, d/2)$ into N pieces and takes the limit $N \to \infty$. As a result, functional derivatives replace ordinary ones in the final expressions.] From the structure of the above equation it seems natural to try a perturbation expansion in the dimensionless parameter V_1/E:

$$\phi_n(x) = \sum_{k \geqslant 0} \left(\frac{V_1}{E}\right)^k \phi_n^{(k)}(x) \tag{B.1}$$

This leads to the recursive perturbation scheme

$$[H_0 - (E + n\hbar\omega)]\,\phi_n^{(k)} = -S(x)\frac{E}{2}\,[\phi_{n+1}^{(k-1)} + \phi_{n-1}^{(k-1)}] \tag{B.2}$$

with "initial" condition

$$H_0\phi_n^{(0)} = (E + n\hbar\omega)\,\phi_n^{(0)}$$

which for our case specializes to

$$\phi_n^{(0)} = \phi_0\delta_{n,0} \tag{B.3}$$

To $O(V_1^0)$ we have the static, unperturbed problem $H_0\phi_0 = E\phi_0$ with solution

$$\begin{aligned} &\text{I:} && e^{ikx} + B_0 e^{-ikx} \\ &\text{II:} && \Xi_0 + \Gamma_0 \\ &\text{III:} && A_0 e^{ikx} \end{aligned} \tag{B.4}$$

where I–III refer to the different regions of space in Fig. 1. The two linearly independent solutions of the unperturbed problem in region II are Ξ_0 and Γ_0. The amplitudes absorbed in Ξ_0 and Γ_0 are, like B_0 and A_0, determined by the boundary conditions.

B.2. Proof That the Scheme Is Well Behaved

Inhomogeneous equations like (B.2) are only meaningful provided that the inhomogeneity is orthogonal to the solutions of the corresponding homogeneous equation. In our case, this translates to a requirement that, in region II, the inhomogeneity has the form

$$\phi_{n+1}^{(k-1)} + \phi_{n-1}^{(k-1)} = \sum_{p \neq 0} \left(\alpha_{n,p}^{(k-1)}\Xi_{n+p} + \beta_{n,p}^{(k-1)}\Gamma_{n+p}\right) \tag{B.5}$$

where Ξ_m and Γ_m are suitably normalized solutions of $[H_0 - (E + m\hbar\omega)]\,\Psi = 0$, and the α's and β's are constants. We therefore need a general proof that, in our perturbation scheme, the dangerous terms Ξ_n and Γ_n are missing on the right-hand side of (B.5).

With the zeroth-order solution given as $\phi_0 = \Xi_0 + \Gamma_0$, the structure of the iteration scheme (B.2) is shown in Fig. 5. The node (n, k) represents the wave function $\phi_n^{(k)}$, i.e., the $O(V_1^k)$ contribution to the nth sideband. According to (B.2), the contribution at each node consists of a particular solution generated by the two neighboring nodes at the previous level, plus a homogeneous part generated at the node itself (with coefficients determined by the particular solution and the matching conditions). What one needs to prove is that a homogeneous solution at (p, q) does not generate a corresponding inhomogeneity in the equation for the contribution at $(p, q + 2l)$.

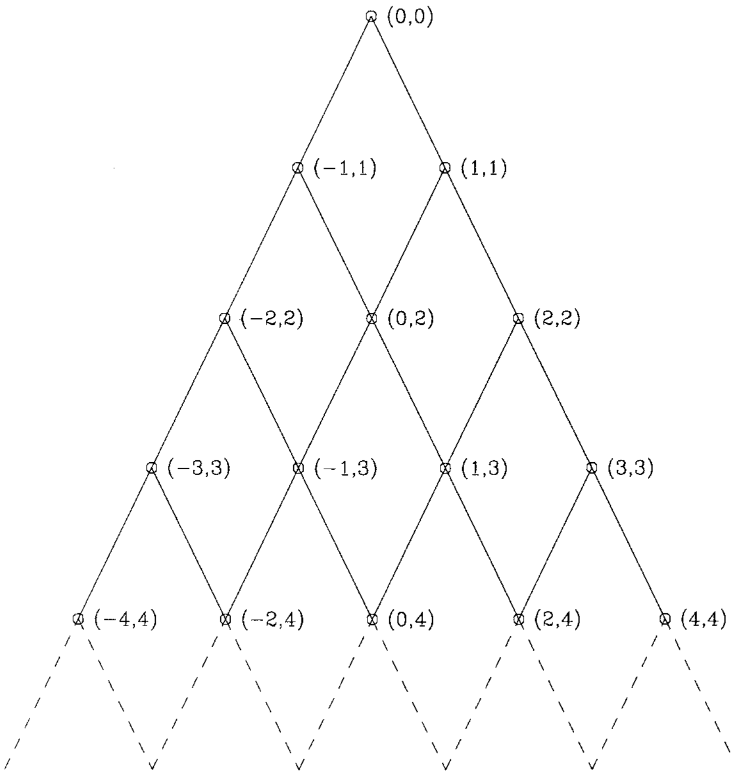

Fig. 5. Structure of the perturbation scheme. The nodes (n, k) represent the wave functions $\phi_n^{(k)}$, i.e., the $O(V_1^k)$ contribution to the nth sideband.

First note that no dangerous inhomogeneity exists in the equation for $\phi_0^{(2)}$. This follows from the fact that (B.2) gives the particular solution $\phi_{\pm 1}^{(1)p}$ as $\phi_{\pm 1}^{(1)p} = \pm(E/2\hbar\omega)\,\phi_0^{(0)}$, and thus $\phi_1^{(1)p} + \phi_{-1}^{(1)p} = 0$. We use this as a basis for a proof by induction. Assume that no term $\sim \phi_0$ exists in the equation for $\phi_0^{(2k)}$ (true for $k = 1$). We proceed to show that it must be true also for the equation for $\phi_0^{(2k+2)}$. The relevant inhomogeneity is the sum of the ϕ_0 contributions at the nodes $(-1, 2k+1)$ and $(1, 2k+1)$. The first is generated by a sum over paths down the left half of the network, starting at $(0, 0)$ and terminating at $(-1, 2k+1)$. By assumption, these paths cannot pass through points with $n = 0$ below the source $(0, 0)$. Since $n < 0$ at every node, each step [see Eq. (B.2)] carries a minus sign. For every path, the number of steps from $(0, 0)$ to $(-1, 2k+1)$ is $2k+1$, so that all contributions from paths on the left are negative. Conversely, all paths from $(0, 0)$ to $(1, 2k+1)$ are to the right of $n = 0$ and give positive contributions. By symmetry, every path to the left has a mirror image to the right. Consequently, the sum over all contributions vanishes. This proves that the homogeneous solution at (0, 0) does not generate a corresponding inhomogeneity in the equation at $(0, 2k)$. But the above argument is independent of the location of the source. It therefore also shows that the homogeneous contribution at (p, q) does not generate a corresponding inhomogeneity at $(p, q+2l)$. This completes the proof that the perturbation scheme is well behaved.

B.3. The ω Expansion

We now return to (B.2) with the aim to understand the low-frequency behavior of

$$[H_0-(E\pm\hbar\omega)]\,\phi^{(1)}_{\pm 1}=-S(x)\frac{E}{2}\phi_0 \tag{B.6}$$

The solutions are

$$\begin{aligned}
&\text{I:} && B^{(1)}_{\pm 1}e^{-ik_{\pm 1}x}\\
&\text{II:} && \phi^{(1)}_{\pm 1}=\phi^{(1)\mathrm{h}}_{\pm 1}+\phi^{(1)\mathrm{p}}_{\pm 1}=C^{(1)}_{\pm 1}\Xi_{\pm 1}+D^{(1)}_{\pm 1}\Gamma_{\pm 1}\pm\frac{1}{2}\frac{E}{\hbar\omega}\phi_0\\
&\text{III:} && A^{(1)}_{\pm 1}e^{ik_{\pm 1}x}
\end{aligned} \tag{B.7}$$

Here $\Xi_{\pm 1}$ and $\Gamma_{\pm 1}$ are the homogeneous solutions to the stationary problem under the barrier, at energies $E\pm\hbar\omega$. The particular solutions $\phi^{(1)\mathrm{p}}_{\pm 1}$ are found by inspection. If we also take $\hbar\omega/E$ to be a small parameter, we can expand around the unperturbed problem at energy E. Since $\phi^{(1)\mathrm{p}}_{\pm 1}=O[(\hbar\omega/E)^{-1}]$, the expansion of the coefficients must be

$$A^{(1)}_{\pm 1}=\left(\frac{\hbar\omega}{E}\right)^{-1}A^{(1,-1)}_{\pm 1}+A^{(1,0)}_{\pm 1}+\frac{\hbar\omega}{E}A^{(1,1)}_{\pm 1}+\cdots \tag{B.8}$$

and similarly for B, C, and D. Furthermore, one has

$$\Xi_n=\Xi_0+nE\frac{\hbar\omega}{E}\frac{\partial\Xi_0}{\partial E}+\cdots \tag{B.9}$$

wth an analogous expansion for Γ_n. The wave functions in regions I and III can be explicitly expanded

$$e^{\pm ik_nx}=e^{\pm ikx}\left[1\pm ikx\frac{n}{2}\frac{\hbar\omega}{E}-(k^2x^2\pm ikx)\frac{n^2}{8}\left(\frac{\hbar\omega}{E}\right)^2+\cdots\right] \tag{B.10}$$

In expanded form, (B.7) reads

$$\begin{aligned}
&\text{I:} && \left(\frac{\hbar\omega}{E}\right)^{-1} B_{\pm 1}^{(1,-1)} e^{-ikx} + \left(B_{\pm 1}^{(1,0)} \pm \frac{1}{2} ikx B_{\pm 1}^{(1,-1)}\right) e^{-ikx} + \cdots \\
&\text{II:} && \left(\frac{\hbar\omega}{E}\right)^{-1} \left[\left(C_{\pm 1}^{(1,-1)} \pm \frac{1}{2}\right) \Xi_0 + \left(D_{\pm 1}^{(1,-1)} \pm \frac{1}{2}\right) \Gamma_0\right] \\
& && + \left[C_{\pm 1}^{(1,0)} \Xi_0 + D_{\pm 1}^{(1,0)} \Gamma_0 \pm C_{\pm 1}^{(1,-1)} E \frac{\partial \Xi_0}{\partial E} \pm D_{\pm 1}^{(1,-1)} E \frac{\partial \Gamma_0}{\partial E}\right] + \cdots \\
&\text{III:} && \left(\frac{\hbar\omega}{E}\right)^{-1} A_{\pm 1}^{(1,-1)} e^{ikx} + \left(A_{\pm 1}^{(1,0)} \pm \frac{1}{2} ikx A_{\pm 1}^{(1,-1)}\right) e^{ikx} + \cdots
\end{aligned} \tag{B.11}$$

In (B.11), the matching conditions at $x = \pm d/2$ must be applied order by order in $\hbar\omega/E$. To $O[(\hbar\omega/E)^{-1}]$, the resulting system of homogeneous, linearly independent equations for the coefficients has the trivial solution only:

$$\begin{aligned}
A_{\pm 1}^{(1,-1)} &= B_{\pm 1}^{(1,-1)} = 0 \\
C_{\pm 1}^{(1,-1)} &= D_{\pm 1}^{(1,-1)} = \mp 1/2
\end{aligned} \tag{B.12}$$

in agreement with the result in Section 2 that the $\omega \to 0$ limit is finite. Using (B.12), one finds that the $O[(\hbar\omega/E)^0]$ wave functions simplify to

$$\begin{aligned}
&\text{I:} && B_{\pm 1}^{(1,0)} e^{-ikx} \\
&\text{II:} && C_{\pm 1}^{(1,0)} \Xi_0 + D_{\pm 1}^{(1,0)} \Gamma_0 - \frac{1}{2} E \frac{\partial \phi_0}{\partial E} \\
&\text{III:} && A_{\pm 1}^{(1,0)} e^{ikx}
\end{aligned} \tag{B.13}$$

where the four coefficients are determined by the boundary conditions. Knowledge of the solution to the unperturbed problem thus enables us to calculate corrections due to the perturbation.

It is not entirely trivial to see how the simple result of Section 2 for the $\omega \to 0$ limit follows from (B.13). We first note that the *form* of the wave function in the barrier only depends on the difference $V(x) - E$. As far as the form is concerned, a small increase δE in E is equivalent to a small *uniform* decrease δV in $V(x)$. One could therefore be tempted to replace $\partial \phi_0/\partial E$ in (B.13) by $-\partial \phi_0/\partial V$, where differentiation is with respect to a uniform shift in $V(x)$. This would not be correct, however. The *amplitudes* of the wave function inside the barrier are coupled to the functions outside through the matching conditions at $x = \pm d/2$. We therefore separate prefactors from functions, ξ and γ, with normalization fixed by a given value at, say, $x=0$, and write

$$\phi_0=\Xi_0+\Gamma_0=C(E, V-E)\,\xi(x; V-E)+D(E, V-E)\,\gamma(x; V-E) \quad \text{(B.14)}$$

so that

$$\frac{\partial\phi_0}{\partial E}=-\left(\frac{\partial\phi_0}{\partial V}\right)_E+\frac{1}{C}\left(\frac{\partial C}{\partial E}\right)_{V-E}\Xi_0+\frac{1}{D}\left(\frac{\partial D}{\partial E}\right)_{V-E}\Gamma_0 \quad \text{(B.15)}$$

The second line in (B.13) can therefore be written as

$$\begin{aligned}\text{II:}\quad &\left[C_{\pm1}^{(1,0)}-\frac{E}{2C}\left(\frac{\partial C}{\partial E}\right)_{V-E}\right]\Xi_0+\left[D_{\pm1}^{(1,0)}-\frac{E}{2D}\left(\frac{\partial D}{\partial E}\right)_{V-E}\right]\Gamma_0\\ &+\frac{1}{2}E\left(\frac{\partial\phi_0}{\partial V}\right)_E \end{aligned} \quad \text{(B.16)}$$

With the form (B.16) in region II, we can now compare (B.13) with the static solution (B.4), differentiated with respect to a uniform change in V, at constant E:

$$\text{I:}\quad \frac{\partial B_0}{\partial V}e^{-ikx}, \qquad \text{II:}\quad \frac{\partial\phi_0}{\partial V}, \qquad \text{III:}\quad \frac{\partial A_0}{\partial V}e^{ikx} \quad \text{(B.17)}$$

Clearly, the square brackets in (B.16) must vanish. That determines the coefficients $C_{\pm1}^{(1,0)}$ and $D_{\pm1}^{(1,0)}$. What remains is, apart from a constant prefactor, $\partial\phi_0/\partial V$ in all three regions, in agreement with the results of Section 2. Reinstating the factor V_1/E from (B.1), one finds, in particular, the transmitted sidebands in the $\omega\to 0$ limit as

$$A_{\pm1}^{(0)}=\frac{V_1}{E}A_{\pm1}^{(1,0)}=\frac{V_1}{2}\frac{\partial A_0}{\partial V} \quad \text{(B.18)}$$

which is a special case of (2.13), to $O(V_1)$.

B.4. The $O(\omega)$ Terms

Equation (B.11), continued to linear order in ω, and with $O(\omega^{-1})$ and $O(\omega^0)$ coefficients now determined, provides the basis for a calculation of the coefficients $A_{\pm1}^{(1,1)}$ etc. The computation is straightforward, but tedious, and the general results are complicated. We shall not quote them here, since their detailed structure is not particularly illuminating. Three remarks are nevertheless in order: (i) The results for $C_{\pm1}^{(1,0)}$ and $D_{\pm1}^{(1,0)}$ from (B.16) already indicate that the $O(\omega)$ terms *cannot* be obtained by differentiations of the stationary solution (B.4) with respect to V alone. A combination of V derivatives and E derivatives is needed. (ii) In fact, the $O(\omega)$ terms cannot even be obtained from the (forward) scattering state (B.4) alone. Also, the general solution of the (backward) scattering problem, with incoming particles from the right, is needed to construct the $O(\omega)$ terms. (iii) It is not surprising that new physics enters in higher order terms. A striking illustration of this is provided by the dynamical localization effect: With $\hbar\omega > E$, localized states, in which the particles remain for a finite time only, exist already to $O(V_1)$, as pointed out by Büttiker and Landauer.[3]

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